Matrix Algebra

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Matrix algebra expresses numbers as collections or "containers" (i.e., [...]) of numbers to more conveniently work with many of them at once. These are some (very) basic notes describing matrix algebra and its implementation in `R`. For more details, see Hansen's (2022) Appendix A.

Defining Scalars, Vectors, and Matrices

A **scalar** is a single number and is denoted by lowercase italics (e.g.):

$$a = 6.$$

```
a<-6
```

A **vector** is a collection of numbers in a single dimension and is denoted by bold lowercase italics. A vector of length n has n **rows** (e.g.):

$$\underset{(n=4)}{\boldsymbol{b}} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 0 \end{bmatrix}.$$

```
b<-matrix(c(3,2,4,0),nrow=4)
```

A vector of width k has k **columns** (e.g.):

$$\underset{(k=4)}{\boldsymbol{c}} = \begin{bmatrix} 7 & 8 & 1 & 2 \end{bmatrix}.$$

```
c<-matrix(c(7,8,1,2),ncol=4)
```

A **column vector** is a vector that can be characterized as a column of numbers, where $k = 1$ (e.g., $\boldsymbol{b}$).

A **row vector** is a vector that can be characterized as a row of numbers, where $n = 1$ (e.g., $\boldsymbol{c}$).

A **matrix** is a collection of numbers in two dimensions and is denoted by bold uppercase italics. Matrices have n rows and k columns (e.g.):

$$\underset{(n=2,\ k=2)}{\boldsymbol{D}} = \begin{bmatrix} 9 & 2 \\ 3 & 0 \end{bmatrix}.$$

```
D<-matrix(c(9,2,
            3,0),nrow=2,ncol=2,byrow=T)
#I like to enter the matrix data as I want it to look, for which "byrow" is useful
```

Matrices are always described by their rows first and then by their columns, with dimensions $n \times k$. For example, $\boldsymbol{D}$ is a 2×2 matrix.

Elements are the numbers contained in a matrix or vector and are denoted with subscripts $_{nk}$ corresponding to the row and or column where they are located (e.g.):

$$\boldsymbol{D}_{11} = 9.$$

```
D[1,1]
```

```
## [1] 9
```

Operations on Scalars, Vectors, and Matrices

Scalars can be added, subtracted, multiplied, and divided by each other using familiar algebra (e.g.):

$$e = 1,\ a - e = (6 - 1) = 5.$$

```
e<-1
a-e
```

```
## [1] 5
```

Vectors and matrices can also be added, subtracted, multiplied, and divided by scalars by applying the respective operation on each element of the vector, known as **element-wise** operation (e.g.):

$$\boldsymbol{b} - e = \begin{bmatrix} (3-1) \\ (2-1) \\ (4-1) \\ (0-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix},\ \boldsymbol{D} - e = \begin{bmatrix} (9-1) & (2-1) \\ (3-1) & (0-1) \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 2 & -1 \end{bmatrix}.$$

```
b-e
```

```
##      [,1]
## [1,]    2
## [2,]    1
## [3,]    3
## [4,]   -1
```

```
D-e
```

```
##      [,1] [,2]
## [1,]    8    1
## [2,]    2   -1
```

However, vectors and matrices can only be added or subtracted together when they are of the same dimension. When this is the case, element-wise operation means that the operation only applies to elements in similar locations (e.g.):

$$\boldsymbol{f} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 3 \end{bmatrix},\ \boldsymbol{b} + \boldsymbol{f} = \begin{bmatrix} (3+5) \\ (2+2) \\ (4+0) \\ (0+3) \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 4 \\ 3 \end{bmatrix}.$$

```r
f<-matrix(c(5,2,0,3),nrow=4)
b+f
```

```
##      [,1]
## [1,]    8
## [2,]    4
## [3,]    4
## [4,]    3
```

Multiplying two vectors or matrices together comes with new requirements. In order for two matrices to be **conformable** for multiplication, the first matrix must have the same number of columns as the rows in the second matrix. If this is the case, the resulting matrix from multiplication will consist of elements that are combinations of the multiplied matrices' rows and columns. If $\boldsymbol{A}$ is $n \times k$ and $\boldsymbol{B}$ is $n \times m$, then the resulting matrix will be $n \times m$ where

$$[\boldsymbol{AB}_{ij}] = \sum_{k=1}^{K} [A]_{ik} [B]_{kj}, \ i = 1, ..., n; j = 1, ..., m.$$

For example,

$$\boldsymbol{G} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}, \ \boldsymbol{D} \times \boldsymbol{G} = \begin{bmatrix} (9 \times 3) + (2 \times 4) & (9 \times 2) + (2 \times 1) \\ (3 \times 3) + (0 \times 4) & (3 \times 2) + (0 \times 1) \end{bmatrix} = \begin{bmatrix} 35 & 20 \\ 9 & 6 \end{bmatrix}.$$

```r
G<-matrix(c(3,2,
            4,1),nrow=2,ncol=2,byrow=T)
D%*%G
```

```
##      [,1] [,2]
## [1,]   35   20
## [2,]    9    6
```

Lastly, "dividing" matrices by each other is more complicated. To perform something like this operation, one must multiply by the **inverse** of a matrix. The inverse of a matrix only exists when it is square and of **full rank**, meaning that none of its columns are linear combinations of each other. In other words, its columns must be **linearly independent**, in which case the rank (or number of linearly independent columns) will equal the number of columns of the matrix. For the $n \times k$ matrix $\boldsymbol{A}$ with rank$(\boldsymbol{A}) = k$, the definition of its inverse matrix $\boldsymbol{A}^{-1}$ is

$$\underset{(n\times k)}{\boldsymbol{A}} \underset{(k\times n)}{\boldsymbol{A}^{-1}} = \underset{(k\times n)}{\boldsymbol{A}^{-1}} \underset{(n\times k)}{\boldsymbol{A}} = \boldsymbol{I}_k.$$

That is, an inverse matrix satisfies the condition that multiplying a matrix by its inverse produces the **identity** matrix ($\boldsymbol{I}_k$). The identity matrix is a $k \times k$ **diagonal** matrix with 1's on the diagonal and 0's on the off-diagonal:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

```
#I_4:
I<-diag(4)
```

To find the inverse of a 2×2 matrix $\boldsymbol{A} = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$, the formula is

$$\boldsymbol{A}^{-1} = \det(\boldsymbol{A}) \times \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

where det() is the **determinant**.

```
#for D:
qr(D)$rank
```

```
## [1] 2
```

```
solve(D)
```

```
##      [,1]       [,2]
## [1,]  0.0  0.3333333
## [2,]  0.5 -1.5000000
```

```
#while "solve" does everything for you, the determinant is:
det(D)
```

```
## [1] -6
```

```
#and demonstrating the condition for an inverse stated earlier:
D%*%solve(D)#=I_2
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

Additional Operations and Notes

For scalars, vectors, and matrices, addition and subtraction is **commutative**, meaning (essentially) that it does not matter the order in which the operation occurs. For multiplication (and therefore "division" or inverse multiplication), this is not true for matrices. For two conformable matrices $\boldsymbol{B}$ and $\boldsymbol{A}$, $\boldsymbol{B}$ **post-multiplied** by $\boldsymbol{A}$ results in the matrix $\boldsymbol{BA}$. This is not necessarily equal to $\boldsymbol{B}$ **pre-multiplied** by $\boldsymbol{A}$, which results in the matrix $\boldsymbol{AB}$.

```
#for D and G:
D%*%G
```

```
##      [,1] [,2]
## [1,]   35   20
## [2,]    9    6
```

```r
G%*%D
```

```
##      [,1] [,2]
## [1,]   33    6
## [2,]   39    8
```

The **transpose** of a matrix $\boldsymbol{A}$ is denoted as $\boldsymbol{A}'$. It is like "flipping" the matrix about its diagonal. For 2×2 $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ \boldsymbol{A}' = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

```r
#for D:
t(D)
```

```
##      [,1] [,2]
## [1,]    9    3
## [2,]    2    0
```

tr() is the **trace** of a matrix, defined as the sum of its diagonal elements:

$$\text{tr}(\boldsymbol{A}) = \sum_{i=1}^{k} a_{ii}.$$

```r
#for D:
sum(diag(D))
```

```
## [1] 9
```

A square matrix that is not full rank is called **singular** and is **non-invertable**. If a matrix is full rank and thus **nonsingular**, its inverse exists and is unique.

The inverse of a 1×1 matrix $\boldsymbol{A}$ with element a_{11} is $[\,\frac{1}{a_{11}}\,]$.

A matrix that is full rank will have a determinant greater than 0.

Exercises

Q: Does the following matrix have full rank?

$$\begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$$

A: No, since $\left[\begin{smallmatrix} 6 \\ 2 \end{smallmatrix}\right] = \frac{2}{3} \times \left[\begin{smallmatrix} 9 \\ 3 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 6 \\ 2 \end{smallmatrix}\right] - (\frac{2}{3} \times \left[\begin{smallmatrix} 9 \\ 3 \end{smallmatrix}\right]) = 0$. Also:

```r
det(matrix(c(6,9,
             2,3),nrow=2,byrow=T))
```

```
## [1] 0
```

Q: What is the following?

$$\boldsymbol{H}=\begin{bmatrix}3 & 10\\ 4 & 1\end{bmatrix},\ \boldsymbol{J}=\begin{bmatrix}2 & 0\\ 5 & 7\end{bmatrix},\ (\boldsymbol{H}'\boldsymbol{J})^{-1}=?$$

```r
H<-matrix(c(3,10,
            4,1),nrow=2,byrow=T)
J<-matrix(c(2,0,
            5,7),nrow=2,byrow=T)
solve(t(H)%*%J)
```

```
##             [,1]        [,2]
## [1,] -0.01351351  0.05405405
## [2,]  0.04826255 -0.05019305
```

Bibliography

Hansen, Bruce E. 2022. *Econometrics*. Princeton University Press.